A variable coefficient nonlinear Schrödinger equation with a four-dimensional symmetry group and blow-up of its solutions

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Abstract

A canonical variable coefficient nonlinear Schrödinger equation with a four dimensional symmetry group containing $\mathrm{SL}(2,\mathbb{R})$ group as a subgroup is considered. This typical invariance is then used to transform by a symmetry transformation a known solution that can be derived by truncating its Painlevé expansion and study blow-ups of these solutions in the L_p-norm for $p > 2$, L_∞-norm and in the sense of distributions.

Keywords: $\mathrm{SL}(2,\mathbb{R})$ invariance, variable coefficient nonlinear Schrödinger equation, exact solutions, blow-up

AMS subject Classifications: Primary 35Q55, 35B44, 35B06; Secondary 35A25

1 Introduction

It is well known that the linear heat and Schrödinger equations for $u \in \mathbb{R}$, $\psi \in \mathbb{C}$

$$u_t = u_{xx}, \quad i\psi_t + \psi_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

have isomorphic Lie symmetry groups. The symmetry group with the infinite-dimensional ideal reflecting linearity factored out can be written as a semidirect product of the three-dimensional Heisenberg group H and $\mathrm{SL}(2,\mathbb{R})$

$$G = \mathsf{H} \ltimes \mathrm{SL}(2,\mathbb{R}).$$

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On the other hand, among the modular class of nonlinear Schrödinger equations in one space dimension

$$i\psi_t + \psi_{xx} = F(|\psi|)\psi, \tag{1.1}$$

the only one preserving the symmetry group of the linear Schrödinger equation except with infinite-dimensional symmetry of its linear counterpart is the quintic Schrödinger equation

$$i\psi_t + \psi_{xx} = \lambda|\psi|^4\psi. \tag{1.2}$$

In this article we look at a variable coefficient extension of the one-dimensional cubic NLSE (nonlinear Schrödinger equation)

$$\begin{aligned} &i\psi_t + \psi_{xx} + g(x,t)|\psi|^2\psi + h(x,t)\psi = 0, \\ &g = g_1 + ig_2, \quad h = h_1 + ih_2, \quad g_j, h_j \in \mathbb{R}, \quad j = 1, 2, \quad g_1 \neq 0. \end{aligned} \tag{1.3}$$

Variable coefficient extensions of nonlinear evolution type equations tend to arise in cases when less idealized conditions such as inhomogeneities and variable topographies are assumed in their derivation. The reader is referred to [8] and the references therein for several physically motivated applications. Symmetry classification of (1.4) was given in [4]. We mention that the dimension of the maximal symmetry group is $\dim G = 5$ and is achieved only when the coefficients are constant and additionally $h = 0$ which is reduced to nothing more than the usual NLS equation. The symmetry group is isomorphic to the group of one-dimensional extended Galilei similitude algebra. We are particularly interested in the case when at least the $\mathrm{SL}(2, \mathbb{R})$ invariance is contained in the full symmetry group of (1.3). This is usually referred to as the pseudo-conformal invariance in the context of qualitative analysis of PDEs.

We emphasize that this invariance manifests itself as a subgroup of the full symmetry groups of the NLSE, Davey-Stewartson (DS) equations and their possible generalizations [5] and has been successfully applied to investigate blowup formation in these nonlinear evolution models [2, 7, 1, 3]. A study of self-similar solutions of the pseudo-conformally invariant nonlinear Schrödinger equation can be found in [6]. The point is that when the variable coefficients are allowed in these equations, this typical symmetry is mostly destroyed. Our intention here is to detect the subcases in which such a symmetry remains intact in variable coefficients variants of the NLS equations. This will make it possible to generate new nontrivial solutions from known ones.

We quote the following result from [4] and note that throughout the paper, any solution will be understood as a pointwise solution in the classical sense.

Proposition 1. *Any equation of the form* (1.3) *containing* $\mathrm{SL}(2, \mathbb{R})$ *symmetry group as a subgroup can be transformed by point transformations to the canonical form*

$$i\psi_t + \psi_{xx} + (\epsilon + i\gamma)\frac{1}{x}|\psi|^2\psi + (h_1 + ih_2)\frac{1}{x^2}\psi = 0, \quad x \in \mathbb{R} \setminus \{0\}, \tag{1.4}$$

where $\epsilon = \pm 1$, γ, h_1 *and* h_2 *are arbitrary real constants.*

Note that like all equations in the modular class (1.1), equation (1.3) and also (1.4) are always invariant under the constant change of phase of ψ (gauge-invariance) while leaving the (x, t) coordinates unchanged. We represent the phase and modulus of ψ by ρ, ω writing $\psi = \rho(x,t)\exp(i\omega(x,t))$.

Proposition 2. *The symmetry algebra L of* (1.4) *is four-dimensional and spanned by the vector fields*

$$T = \partial_t, \quad D = 2t\partial_t + x\partial_x - \frac{1}{2}\rho\partial_\rho, \quad C = t^2\partial_t + xt\partial_x - \frac{1}{2}t\rho\partial_\rho + \frac{x^2}{4}\partial_\omega, \quad W = \partial_\omega.$$

The commutators among T, D, C satisfy

$$[T, D] = 2T, \quad [T, C] = D, \quad [D, C] = 2C$$

and W is the center element, namely commutes with all other elements. The Lie algebra L has the direct sum structure

$$L = \mathrm{sl}(2, \mathbb{R}) \oplus \mathbb{R}.$$

The elements T, D, C, W generate time translations, scaling, (pseudo)-conformal and gauge transformations, respectively. A significant consequence of these transformations is the group action on the solutions given by the following.

Proposition 3. *If $\psi_0(x,t)$ is a solution of* (1.4), *then so is*

$$\psi(x,t) = (a+bt)^{-1/2}\,\mathrm{e}^{i\frac{bx^2}{4(a+bt)}}\,\psi_0\left(\frac{x}{a+bt}, \frac{c+dt}{a+bt}\right)$$

for $ad - bc = 1$.

Proof. By exponentiating the infinitesimal generators T, D, C (i.e. solving Cauchy problems for these vector fields) and then composing the corresponding group transformations we find the above $\mathrm{SL}(2, \mathbb{R})$ action on the solutions. Note that the action corresponding to t generates Möbius transformations of t.

Based on this result, our main purpose is to transform one known solution of the original equation to a more complicated one by the transformations of the $\mathrm{SL}(2, \mathbb{R})$ symmetry and then choose group parameters a, b, c appropriately and pass to limit of the wave function ψ as $t \to T^-$ for some finite time T in the L_p-norm and distributional sense as well.

We now use a truncation approach to obtain a special exact solution of (1.4). We are going to find this special explicit solution by truncating its Painlevé expansion at the first term. For convenience we write (1.4) together with its complex conjugate as the system

$$\begin{aligned} iu_t + u_{xx} + (\epsilon + i\gamma)\frac{1}{x}u^2 v + (h_1 + ih_2)\frac{1}{x^2}u = 0,\\ -iv_t + v_{xx} + (\epsilon - i\gamma)\frac{1}{x}uv^2 + (h_1 - ih_2)\frac{1}{x^2}v = 0. \end{aligned} \tag{1.5}$$

Here u was employed instead of ψ and v denotes its complex conjugate, but in this setting they are viewed as independent functions. We first show that (1.5) does not pass the Painlevé test for PDEs and then proceed to obtain an exact solution afterwards.

A partial differential equation is said to have the Painlevé property if all its solutions are single valued around any non-characteristic movable singularity manifold. If this singularity manifold is denoted by $\Phi(x,t) = 0$ (actually a curve in this case), we shall look for solutions of the system (1.5) in the form of a Laurent expansion and we expand

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t)\Phi^{\alpha+j}(x,t), \quad v(x,t) = \sum_{j=0}^{\infty} v_j(x,t)\Phi^{\beta+j}(x,t), \tag{1.6}$$

where $u_0, v_0 \neq 0$ and $u_j, v_j, \Phi(x,t)$ are analytic functions. α and β are negative integers to be determined from the leading order analysis, so as to ensure absence of essential singularities and branch points in all solutions. For the determination of leading orders α and β, we substitute $u \sim u_0\Phi^{\alpha}$ and $v \sim v_0\Phi^{\beta}$ in (1.5) and see that by balancing the terms of smallest order

$$\alpha + \beta = -2 \tag{1.7}$$

and

$$u_0 v_0 = -\alpha(\alpha-1)\frac{x}{\epsilon + i\gamma}\Phi_x^2 = -\beta(\beta-1)\frac{x}{\epsilon - i\gamma}\Phi_x^2 \tag{1.8}$$

must hold. (1.7) allows the negative integers $\alpha = -1$ and $\beta = -1$ and with these leading orders, (1.8) forces $\gamma = 0$. After determination of the leading orders, we substitute (1.6) into (1.5). Equating to zero the coefficient of Φ^{-3+j}, $j \geq 1, j \in \mathbb{N}$, we arrive at a linear system

$$Q(j)\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} F_j \\ G_j \end{pmatrix} \tag{1.9}$$

from which the coefficients u_j, v_j can be found. Those values of indices j for which $\det Q(j) = 0$ are called *resonances*. In order that the expansion (1.6) includes correct number of arbitrary functions as required by the Cauchy-Kovalewski theorem (where $\Phi(x,t)$ should be one of the arbitrary functions), some consistency conditions at resonances must be satisfied. In the general case (1.3), these constraints force the coefficients to be properly related (See [8] for details). For (1.4), this is not the case, namely it cannot pass the Painlevé test. It thus fails to satisfy the necessary condition for the equation to have Painlevé property.

Notwithstanding this fact, application of the Painlevé expansion to nonintegrable PDEs like (1.4) (or more properly partial integrable) can allow particular explicit solutions to be obtained by truncating the expansion. This approach imposes constraints on the arbitrary functions and the function Φ as a result of compatibility of an overdetermined PDE system.

Before truncating the series (1.6) at some index $j = N$, we first weaken the condition that α and β assumes only integer values. When we solve (1.7) and (1.8) together, we find that for $\gamma \neq 0$

$$\alpha = -1 - i\delta, \quad \beta = -1 + i\delta; \qquad \delta = \frac{-3\epsilon \pm \sqrt{8\gamma^2 + 9}}{2\gamma} \tag{1.10}$$

and (1.8) is equivalent to

$$u_0 v_0 = -\frac{3\delta}{\gamma}\, x\, \Phi_x^2. \tag{1.11}$$

Truncating the Painlevé expansion at the first term ($j = 0$), we assume a solution of the form

$$u(x,t) = u_0(x,t)\Phi(x,t)^{-1-i\delta}, \quad v(x,t) = v_0(x,t)\Phi(x,t)^{-1+i\delta}. \tag{1.12}$$

We substitute these in (1.5) and set the coefficients of the terms $\Phi^{-3\pm i\delta}$, $\Phi^{-2\pm i\delta}$, $\Phi^{-1\pm i\delta}$ equal to zero. The condition at the order $\Phi^{-3\pm i\delta}$ is equivalent to (1.11) and terms of order $\Phi^{-2\pm i\delta}$, $\Phi^{-1\pm i\delta}$ disappear if Φ, u_0, v_0 are chosen to be

$$\Phi(x,t) = \left(\frac{x}{k_4 t + k_1}\right)^{2/3} + k_2 \tag{1.13}$$

$$u_0(x,t) = A_1 \frac{x^{1/6}}{(k_4 t + k_1)^{2/3}} \exp\left[i\Big(\frac{k_4 x^2}{4(k_4 t + k_1)} + k_3\Big)\right] \tag{1.14}$$

$$v_0(x,t) = A_2 \frac{x^{1/6}}{(k_4 t + k_1)^{2/3}} \exp\left[-i\Big(\frac{k_4 x^2}{4(k_4 t + k_1)} + k_3\Big)\right] \tag{1.15}$$

for arbitrary real constants A_1, A_2 and $k_1, ..., k_4$ with the constraints $h_1 = 5/36$ and $h_2 = 0$. We require to have $v = u^*$, which implies $A_1 = A_2$. Let us rename this constant A. When we check the condition (1.11), we find that $A^2 = \frac{-4\delta}{3\gamma}$. Since this square must be positive, it is necessary that

$$\frac{\delta}{\gamma} = \frac{-3\epsilon \pm \sqrt{8\gamma^2 + 9}}{2\gamma^2} < 0. \tag{1.16}$$

For both $\epsilon = \pm 1$, it is seen that the minus sign must be picked in the formula for δ of (1.10). Having found a consistent truncation, we can write the solution to (1.4) as

$$u(x,t) = \frac{A x^{1/6}}{x^{2/3} + k_2 (k_4 t + k_1)^{2/3}} \exp\left[i\left(\frac{k_4 x^2}{4(k_4 t + k_1)} - \delta \ln\left(\frac{x^{2/3}}{(k_4 t + k_1)^{2/3}} + k_2\right) + k_3\right)\right]. \tag{1.17}$$

Choosing $k_4 = 0$, we obtain the stationary solution

$$u(x) = \frac{A x^{1/6}}{x^{2/3} + k_1^{2/3} k_2} \exp\left[i\left(-\delta \ln(x^{2/3} + k_1^{2/3} k_2) + k_3\right)\right], \tag{1.18}$$

where

$$h_1 = \frac{5}{36}, \quad h_2 = 0, \quad A = (-\frac{4\delta}{3\gamma})^{1/2}, \quad \delta = \frac{-3\epsilon - \sqrt{8\gamma^2 + 9}}{2\gamma}, \tag{1.19}$$

and k_1, k_2, k_3 (k_3 is relabelled) are arbitrary real constants.
We summarize:

Proposition 4. *The following solves equation* (1.4) *for arbitrary constants k_1, k_2, k_3 and for the parameters $h_1 = 5/36$, $h_2 = 0$*

$$u(x,t) = A\,\frac{x^{1/6}}{x^{2/3} + k_1^{2/3} k_2} \exp\left[i\left(-\delta \ln(x^{2/3} + k_1^{2/3} k_2) + k_3\right)\right], \tag{1.20}$$

where $A = (-\frac{4\delta}{3\gamma})^{1/2}$ and $\delta = \frac{-3\epsilon - \sqrt{8\gamma^2+9}}{2\gamma}$.

2 Transforming solutions by $\mathrm{SL}(2,\mathbb{R})$ group

Blowup in the L_p, L_∞ norms and in the distributional sense

Now we would like to illustrate how the $\mathrm{SL}(2,\mathbb{R})$ group action can be useful in establishing blow-up profiles of initial value problems for variable coefficient NLS equations just as they were used for their constant coefficient counterparts.

Let $u(x)$ be the stationary solution to (1.4), defined by (1.18)-(1.19). We set $\psi_0(x) := u(x)$ and use Proposition 3. By this proposition for arbitrary $a, b \in \mathbb{R}$

$$\psi(x,t) = (a+bt)^{-1/2} \exp\left(\frac{ibx^2}{4(a+bt)}\right) \psi_0\left(\frac{x}{a+bt}\right) \tag{2.1}$$

is also a solution to equation (1.4).

We can assume $a > 0, b < 0$ and denote $\varepsilon := a + bt = b(t - \frac{-a}{b}) = b(t - T)$, where $T = -\frac{a}{b} > 0$. Hence, $t \to T^- \Leftrightarrow \varepsilon \to 0^+$. By using this notation we can write solution (2.1) in the form

$$\psi_\varepsilon(x) = \varepsilon^{-1/2} \exp\left(\frac{ibx^2}{4\varepsilon}\right) \psi_0\left(\frac{x}{\varepsilon}\right).$$

We are going to show that these solutions will blowup in the L_p-norm when $p > 2$, L_∞-norm and in the sense of generalized functions, respectively.

Note: In the following theorems, we do not impose an initial condition but rather we limit to the one as dictated by the solution (2.1) in the form

$$\psi(x,0) = \frac{1}{\sqrt{a}} \exp\left(\frac{ibx^2}{4a}\right) \psi_0\left(\frac{x}{a}\right).$$

Also, for a blow-up at some finite time we have to fix b figuring in (2.1).

L_p**-blow-up solutions.**

Theorem 1. *For any $T>0$ there is a solution $\psi(x,t)$ to equation* (1.4) *such that*

$$\lim_{t\to T^-}\|\psi(x,t)\|_p=+\infty \quad \textit{for all } p>2, \tag{2.2}$$

where $\|\psi(x,t)\|_p=\left(\int_{-\infty}^{+\infty}|\psi|^p\,dx\right)^{1/p}$.

Proof. Let $T>0$ be a finite time. We can always arrange two numbers $a>0$ and $b<0$ such that $T=-\frac{a}{b}$. Setting these numbers in (2.1) we get the function $\psi(x,t)$ which will be instrumental in the proof. We rewrite this function, as given above, in the form

$$\psi_\varepsilon(x)=\varepsilon^{-1/2}\exp\left(\frac{ibx^2}{4\varepsilon}\right)\psi_0\left(\frac{x}{\varepsilon}\right).$$

Then we have

$$\lim_{\varepsilon\to 0^+}\|\psi_\varepsilon(x)\|_p=\lim_{t\to T^-}\|\psi(x,t)\|_p. \tag{2.3}$$

By the definition of $\psi_0(x)$ we have

$$\psi_0\left(\frac{x}{\varepsilon}\right)=\frac{A\varepsilon^{-\frac{1}{6}}x^{\frac{1}{6}}}{\varepsilon^{-\frac{2}{3}}x^{\frac{2}{3}}+C}\exp\bigl(i\varphi(x,\varepsilon)\bigr),$$

where $\varphi(x,\varepsilon)=-\delta\ln\Bigl(\left(\frac{x}{\varepsilon}\right)^{\frac{2}{3}}+k_1^{\frac{2}{3}}k_2+k_3\Bigr)$, $C=k_1^{\frac{2}{3}}k_2$ and we choose $C>0$. Thus,

$$\psi_\varepsilon(x)=\frac{Ax^{\frac{1}{6}}}{x^{\frac{2}{3}}+\varepsilon^{\frac{2}{3}}C}\exp\Bigl[i\Bigl(\frac{bx^2}{4\varepsilon}+\varphi(x,\varepsilon)\Bigr)\Bigr],$$

and

$$\int_{-\infty}^{\infty}|\psi_\varepsilon(x)|^p dx=2\int_0^{\infty}\frac{A^p x^{p/6}}{\bigl(x^{2/3}+\varepsilon^{2/3}C\bigr)^p}\,dx.$$

By (2.3) $\lim_{t\to T^-}\|\psi(x,t)\|_p=+\infty$ if and only if:

i) $\int_0^\infty|\psi_\varepsilon(x)|^p$ is finite for all $p>2$ and $\varepsilon>0$,

ii) $\lim_{\varepsilon\to 0^+}\int_0^\infty|\psi_\varepsilon(x)|^p dx=\infty$.

Let us substitute $x=\varepsilon y$. Then

$$\begin{aligned}\int_{-\infty}^{\infty}|\psi_\varepsilon(x)|^p dx &= 2A^p\int_0^\infty\frac{\varepsilon^{\frac{p}{6}}y^{\frac{p}{6}}}{\varepsilon^{\frac{2p}{3}}\bigl(y^{\frac{2}{3}}+C\bigr)^p}\varepsilon dy=\frac{2A^p}{\varepsilon^{(p-2)/2}}\int_0^\infty\frac{y^{\frac{p}{6}}}{\bigl(y^{\frac{2}{3}}+C\bigr)^p}dy=\\ &=\frac{2A^p}{\varepsilon^{(p-2)/2}}\Bigl[\int_0^1\frac{y^{\frac{p}{6}}}{\bigl(y^{\frac{2}{3}}+C\bigr)^p}dy+\int_1^{+\infty}\frac{y^{\frac{p}{6}}}{\bigl(y^{\frac{2}{3}}+C\bigr)^p}dy\Bigr],\end{aligned} \tag{2.4}$$

where $\int_0^1 \frac{y^{\frac{p}{6}}}{\left(y^{\frac{2}{3}}+C\right)^p}dy$ is convergent by continuity of the function. On the other hand,

$$\frac{y^{\frac{p}{6}}}{\left(y^{\frac{2}{3}}+C\right)^p} \leq \frac{y^{\frac{p}{6}}}{y^{\frac{2p}{3}}} = \frac{1}{y^{\frac{p}{2}}}.$$

Consequently, $\int_1^\infty \frac{y^{\frac{p}{6}}}{\left(y^{\frac{2}{3}}+C\right)^p}dy \leq \int_1^\infty \frac{1}{y^{\frac{p}{2}}}dy$ implies that the integral $\int_1^\infty \frac{y^{\frac{p}{6}}}{\left(y^{\frac{2}{3}}+C\right)^p}dy$ is convergent for all $p > 2$. Hence, $\int_0^\infty |\psi_\varepsilon(x)|^p dx$ is convergent for $p > 2$. Finally, by taking the limit we find

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} |\psi_\varepsilon(x)|^p dx = \lim_{\varepsilon\to 0^+} \frac{2A^p}{\varepsilon^{(p-2)/2}} \int_0^\infty \frac{y^{\frac{p}{6}}}{\left(y^{\frac{2}{3}}+C\right)^p}dy = \infty.$$

Blow-up solutions in L_∞- norms. In the following theorem we prove that the above defined solutions $\psi_\varepsilon(x)$ will blowup in L_∞- norm too.

Theorem 2. *For any $T > 0$ there is a solution $\psi(x,t)$ to equation* (1.4) *such that*

$$\lim_{t\to T^-} \|\psi(x,t)\|_\infty = \infty, \tag{2.5}$$

where $\|\psi(x,t)\|_\infty = \operatorname{ess\,sup}_{x\in[0,\infty)} |\psi(x,t|,\ t < T$.

Proof. Let again $\psi(x,t)$ be the function defined by (2.1) and $\psi_\varepsilon(x)$ be the above defined function. By the construction of $\psi_\varepsilon(x)$

$$\lim_{\varepsilon\to 0^+} \|\psi_\varepsilon(x)\|_\infty = \lim_{t\to T^-} \|\psi(x,t)\|_\infty$$

and therefore we are done if we can show that $\lim_{\varepsilon\to 0^+} \|\psi_\varepsilon(x)\|_\infty = \infty$. We have $|\psi_\varepsilon(x)| = \frac{A|x|^{\frac{1}{6}}}{x^{\frac{2}{3}}+\varepsilon^{\frac{2}{3}}C}$. Let $A = C = 1$. $|\psi_\varepsilon(x)|$ is an even function so that we can restrict ourselves to the interval $[0,\infty)$. Since $|\psi_\varepsilon(x)|$ is continuous on $[0,\infty)$, $|\psi_\varepsilon(0)| = 0$ and $\lim_{x\to\infty} |\psi_\varepsilon(x)| = 0$, then there exists $x_0 \in (0,\infty)$ such that

$$\|\psi_\varepsilon(x)\|_\infty = \operatorname{ess}\sup_{x\in[0,\infty)} |\psi_\varepsilon(x| = \max_{x\in[0,\infty)} |\psi_\varepsilon(x)| = \frac{x_0^{\frac{1}{6}}}{x_0^{\frac{2}{3}} + \varepsilon^{\frac{2}{3}}C}.$$

A simple computation yields $x_0 = \frac{\varepsilon}{\sqrt{27}}$. Therefore,

$$\|\psi_\varepsilon(x)\|_\infty = C\frac{\varepsilon^{\frac{1}{6}}}{\varepsilon^{\frac{2}{3}}} = \frac{C}{\sqrt{\varepsilon}}.$$

Consequently,

$$\lim_{\varepsilon\to 0^+} \|\psi_\varepsilon(x)\|_\infty = \infty.$$

δ-Blowup solutions in the sense of generalized functions. Let $D = C_0^\infty(0,\infty)$ be the space of infinitely differentiable functions with compact support in $(0,\infty)$. The dual of D is called the space of generalized functions and is denoted by D'.

Definition 1. *Let $f,\ f_n \in D'$. We say that the sequence f_n converges to f if and only if $\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$ for all $\varphi \in D$, where $\langle f, \varphi\rangle$ denotes the value of the functional f at φ.*

Now we present δ-blowup solutions in the sense of generalized functions.

Theorem 3. *For all $p > 2$*

$$\varepsilon^{(p-2)/2}|\psi_\varepsilon(x)|^p \to K\delta(x) \quad \text{as } \varepsilon \to 0^+,$$

where $K = A^p \int_{-\infty}^{\infty} \frac{|y|^{\frac{p}{6}}}{\left(y^{\frac{2}{3}}+C\right)^p} dy$, $C > 0$ and $\delta(x)$ denotes the Dirac distribution at the origin.

Proof. By Definition 1 we have to show that

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} \varepsilon^{(p-2)/2}|\psi_\varepsilon(x)|^p\varphi(x)dx = K\varphi(0) \quad \text{for all } \varphi \in D.$$

Evidently,

$$\varepsilon^{(p-2)/2}\int_{-\infty}^{\infty} |\psi_\varepsilon(x)|^p\varphi(x)\,dx = \varepsilon^{(p-2)/2}\int_{-\infty}^{\infty} \frac{A^p|x|^{p/6}}{\left(x^{2/3}+\varepsilon^{2/3}C\right)^p}\varphi(x)\,dx.$$

By setting the substitution $x = \varepsilon y$ we obtain

$$\begin{aligned}\varepsilon^{(p-2)/2}\int_{-\infty}^{\infty} |\psi_\varepsilon(x)|^p\varphi(x)\,dx &= \varepsilon^{(p-2)/2}\int_{-\infty}^{\infty} \frac{\varepsilon^{\frac{p}{6}}A^p|y|^{p/6}}{\varepsilon^{\frac{2p}{3}}\left(y^{2/3}+C\right)^p}\varphi(\varepsilon y)\,\varepsilon dy \\ = \varepsilon^{(p-2)/2}\int_{-\infty}^{\infty} \frac{A^p|y|^{p/6}}{\varepsilon^{(p-2)/2}\left(y^{2/3}+C\right)^p}\varphi(\varepsilon y)\,dy &= \int_{-\infty}^{\infty} \frac{A^p|y|^{p/6}}{\left(y^{2/3}+C\right)^p}\varphi(\varepsilon y)\,dy.\end{aligned} \tag{2.6}$$

Let $f_\varepsilon(y) := \frac{A^p|y|^{p/6}}{\left(y^{2/3}+C\right)^p}\varphi(\varepsilon y)$. The sequence $f_\varepsilon(y)$ satisfies the following two conditions:

i) $|f_\varepsilon(y)| \le C_\varepsilon \frac{A^p|y|^{p/6}}{\left(y^{2/3}+C\right)^p} \in L_1(-\infty,\infty)$ if $p > 2$,

ii) $\lim_{\varepsilon\to 0^+} f_\varepsilon(y) = \frac{A^p|y|^{p/6}}{\left(y^{2/3}+C\right)^p}\varphi(0)$.

Then by Lebesgue's dominated convergence theorem we obtain

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} f_\varepsilon(y)dy = \varphi(0)\int_{-\infty}^{\infty} \frac{A^p|y|^{p/6}}{\left(y^{2/3}+C\right)^p}\,dy = K\varphi(0).$$

By Definition 1, this implies that

$$\varepsilon^{(p-2)/2}|\psi_\varepsilon(x)|^p \to K\delta(x),$$

as $\varepsilon \to 0^+$.

Remark: The above argument indicates that the singular behavior of the solution is even much worse than in the usual distributional sense as was done in Ref. [7].

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